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## A note on digitized angles

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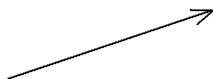
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### SUMMARY

**We study the configurations of pixels that occur when two digitized straight lines meet each other. The exact number of different configurations is calculated when the lines have rational slopes. This theory helps to explain the empirically observed phenomenon that the two “halves” of an arrowhead don’t look the same.**

**KEY WORDS** Pixels (pels), Digitized images, Arrows, Angle bisection, Bresenham algorithm

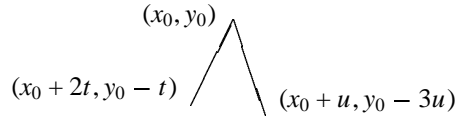
About ten years ago I was supervising the Ph.D. thesis of Chris Van Wyk [4], which introduced the IDEAL language for describing pictures [5]. Two of his example illustrations showed arrows constructed from straight lines something like this:



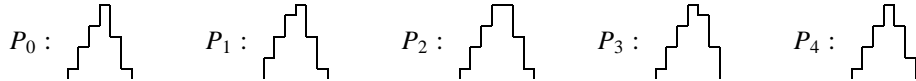
When I looked at them, I was sure that there must be a bug either in IDEAL or in the TROFF processor that typeset the IDEAL output, because the long shafts of the arrows did not properly bisect the angle made by the two short lines of the arrowheads. The shafts seemed to be drawn one pixel too high or too low. Chris spent many hours together with Brian Kernighan trying to find out what was wrong, but no errors could be pinned down. Eventually his thesis was printed on a high-resolution phototypesetter, and the problem became much less noticeable than it had been on the laser-printed proofs. There still was a glitch, but I decided not to hold up Chris’s graduation for the sake of a misplaced pixel.

I remembered this incident at the end of 1983, when I was getting ready to write a new version of the METAFONT system for digital art [3]. I didn’t want my system to have such a flaw. But to my surprise, I learned that the problem is actually unavoidable in raster output: It is almost impossible to bisect a digitized angle exactly, except in very special circumstances. The two “halves” of the angle will necessarily appear somewhat different from each other, unless the resolution is quite high. Therefore Van Wyk (and Kernighan) were vindicated. Similar problems are bound to occur in MacDraw and in any other drawing package.

For example, one of the things I noticed was the following curious fact. Consider the  $45^\circ$  angle that is made when a straight line segment of slope 2 comes up to a point  $(x_0, y_0)$  and then goes down along another line of slope  $-3$ :



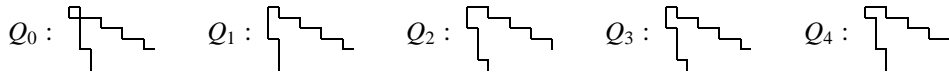
If we digitize this angular path, the result will take one of five different shapes, depending on the value of the intersection point  $(x_0, y_0)$ , whose coordinates are not necessarily integers. The possibilities are



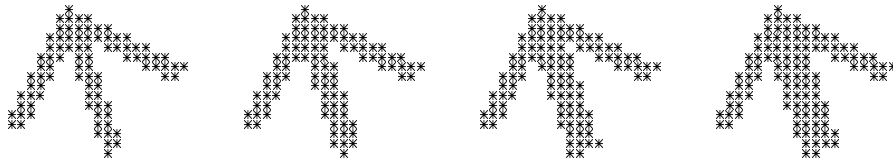
If this  $45^\circ$  angle provides the left half of an arrowhead, the right half of the arrowhead will be a  $45^\circ$  angle made by a line of slope  $-3$  meeting a line of slope  $-1/2$ ,



For this angle there are, similarly, five possibilities after digitization, namely



To complete the arrowhead, we should match the left angle  $P_i$  with an appropriate  $Q_j$ . But none of the  $Q$ 's has the same shape as any of the  $P$ 's. And this is the point: Human eyes tend to judge the magnitude of an angle by its appearance at the tip. By this criterion, some of these angles appear to be quite a bit larger than others (except at high resolutions). Hence it is not surprising that a correctly drawn angle of type  $P$  would appear to be unequal to a correctly drawn angle of type  $Q$ , even though both angles would really be  $45^\circ$  when drawn with infinite resolution. (The patterns of white pixels, not black pixels, are the source of the inconsistency.) Here, for example, are four quite properly digitized arrows with shafts of increasing thickness:



We might also want to know the probability that the digitized shape will be of a particular type  $P_k$ , when the corner point  $(x_0, y_0)$  is chosen at random in the plane. Is one of the patterns more likely to occur than the others? The answer is no, when we use the most natural method of digitization; each  $P_k$  will be obtained with probability  $1/5$ . Similarly, each of the five shapes  $Q_k$  turns out to be equally likely, as  $(x_1, y_1)$  varies.

The main purpose of this note is to prove that the facts just stated are special cases of a general phenomenon:

**Theorem.** *When a line of slope  $a/b$  meets a line of slope  $c/d$  at a point  $(x_0, y_0)$ , the number of different digital shapes it can produce as  $(x_0, y_0)$  varies is  $|ad - bc|$ . Moreover, each of these shapes is equally likely to occur, if  $(x_0, y_0)$  is chosen uniformly in the plane.*

We assume that  $a/b$  and  $c/d$  are rational numbers in lowest terms. Two digitized shapes are considered to be equal if they are identical after translation; rotation and reflection are not allowed.

Before we can prove the theorem, we need to define exactly what it means to digitize a curve. For this, we follow the general idea explained, for example, in Reference [3] (Chapter 24). We consider the plane to be tiled with pixels, which are the unit squares whose corners have integer coordinates. Our goal is to modify a given curve so that it travels entirely on the boundaries between pixels. If the curve is given in parametric form by the function  $z(t) = (x(t), y(t))$  as  $t$  varies, its digitization is essentially defined by the formula

$$\text{round } z(t) = (\text{round } x(t), \text{round } y(t))$$

as  $t$  varies, where  $\text{round}(\alpha)$  is the integer nearest  $\alpha$ .

We need to be careful, of course, when rounding values that are halfway between integers, because  $\text{round}(\alpha)$  is undefined in such cases. Let us assume for convenience that the path  $z(t)$  does not go through any pixel centers; i.e., that  $z(t)$  is never equal to  $(m + \frac{1}{2}, n + \frac{1}{2})$  for integer  $m$  and  $n$ . (Exact hits on pixel centers occur with probability zero, so they can be ignored in the theorem we wish to prove. An infinitesimal shift of the path can be used to avoid pixel centers in general, therefore avoiding the ambiguities pointed out in Bresenham's interesting discussion [1]; but we need not deal with such complications.) Under this assumption, whenever we have  $x(t) = m + \frac{1}{2}$  so that 'round  $x(t)$ ' is ambiguous, the value of  $\text{round } y(t) = n$  will be unambiguous, and we can include the entire line segment from  $(m, n)$  to  $(m + 1, n)$  in the digitized path. Similarly, when  $t$  reaches a value such that  $\text{round } x(t) = m$  but  $\text{round } y(t) = n$  or  $n + 1$ , we include the entire segment from  $(m, n)$  to  $(m, n + 1)$ . This convention defines the desired digitized path,  $\text{round } z(t)$ .

When the path  $z(t)$  returns to its starting point or begins and ends at infinity, without intersecting itself, it defines a region in the plane. The corresponding digitized path,  $\text{round } z(t)$ , also defines a region; and this digitized region turns out to have a simple characterization, when we apply standard mathematical conventions about "winding numbers": *The pixel with corners at  $(m, n)$ ,  $(m + 1, n)$ ,  $(m, n + 1)$ ,  $(m + 1, n + 1)$  belongs to the digitized region defined by  $\text{round } z(t)$  if and only if its center point  $(m + \frac{1}{2}, n + \frac{1}{2})$  belongs to the undigitized region defined by  $z(t)$ .* (This beautiful property of digital curves is fairly easy to verify in simple cases, but a rigorous proof is difficult because it relies ultimately on things like the Jordan Curve Theorem. The necessary details appear in an appendix to John Hobby's thesis [2], Theorem A.4.1.)

Now we are ready to begin proving the desired result. The region defined by an angle at  $(x_0, y_0)$  with lines of slopes  $a/b$  and  $c/d$  can be characterized by the inequalities

$$a(x - x_0) - b(y - y_0) \geq 0; \quad c(x - x_0) - d(y - y_0) \geq 0.$$

(We may need to reverse the signs, depending on which of the four regions defined by two lines through  $(x_0, y_0)$  are assumed to be defined by the given angular path; this can

be done by changing  $(a, b)$  to  $(-a, -b)$  and/or  $(c, d)$  to  $(-c, -d)$ .) This region contains the pixel with lower left corner  $(m, n)$  if and only if

$$a(m + \frac{1}{2} - x_0) - b(n + \frac{1}{2} - y_0) \geq 0; \quad c(m + \frac{1}{2} - x_0) - d(n + \frac{1}{2} - y_0) \geq 0.$$

We can simplify the notation by combining several constants, letting  $\alpha = a(x_0 - \frac{1}{2}) - b(y_0 - \frac{1}{2})$  and  $\beta = c(x_0 - \frac{1}{2}) - d(y_0 - \frac{1}{2})$ :

$$am - bn \geq \alpha; \quad cm - dn \geq \beta.$$

Let  $R(\alpha, \beta)$  be the digitized region consisting of all integer pairs  $(m, n)$  satisfying this condition; these are the pixels in the digitized angle corresponding to  $(x_0, y_0)$ .

As noted above, it is safe to assume that the pixel centers do not exactly touch the lines forming the angle; thus we are free to stipulate that  $am - bn \neq \alpha$  and  $cm - dn \neq \beta$  for all pairs of integers  $(m, n)$ . However, if equality does occur, we might as well define the digitized region  $R(\alpha, \beta)$  by the general inequalities  $am - bn \geq \alpha$  and  $cm - dn \geq \beta$ , as stated, instead of treating this circumstance as a special case. Notice that  $R(\alpha, \beta)$  is equal to  $R(\lceil \beta \rceil)$ ; therefore we can assume that  $\alpha$  and  $\beta$  are integers, in the following discussion.

Another corner point  $(x'_0, y'_0)$  will lead to parameters  $(\alpha', \beta')$  defining another region  $R(\alpha', \beta')$  in the same way. The two regions  $R(\alpha, \beta)$  and  $R(\alpha', \beta')$  have the same shape if and only if one is a translation of the other; i.e.,  $R(\alpha, \beta) \equiv R(\alpha', \beta')$  if and only if there exist integers  $(k, l)$  such that

$$(m, n) \in R(\alpha, \beta) \iff (m - k, n - l) \in R(\alpha', \beta').$$

Our main goal is to prove that the number of distinct region shapes, according to this notion of equivalence, is exactly  $|ad - bc|$ .

**Lemma.** *Let  $\alpha, \beta, \alpha', \beta'$  be integers. Then  $R(\alpha, \beta) \equiv R(\alpha', \beta')$  with respect to slopes  $a/b$  and  $c/d$  if and only if  $\alpha - \alpha' = ka - lb$  and  $\beta - \beta' = kc - ld$  for some integers  $(k, l)$ .*

*Proof.* Assume that  $R(\alpha, \beta) \equiv R(\alpha', \beta')$  with respect to  $a/b$  and  $c/d$ , and let  $(k, l)$  be the corresponding translation amounts. Thus we have

$$\left\{ \begin{array}{l} am - bn \geq \alpha \\ cm - dn \geq \beta \end{array} \right\} \iff \left\{ \begin{array}{l} a(m - k) - b(n - l) \geq \alpha' \\ c(m - k) - d(n - l) \geq \beta' \end{array} \right\}$$

for all integer pairs  $(m, n)$ . Let  $\alpha'' = \alpha' + ka - lb$  and  $\beta'' = \beta' + kc - ld$ , so that

$$\left\{ \begin{array}{l} am - bn \geq \alpha \\ cm - dn \geq \beta \end{array} \right\} \iff \left\{ \begin{array}{l} am - bn \geq \alpha'' \\ cm - dn \geq \beta'' \end{array} \right\}$$

for all integers  $(m, n)$ . This implies that  $\alpha = \alpha''$  and  $\beta = \beta''$ . For if, say, we had  $\alpha < \alpha''$ , we could find integers  $m$  and  $n$  such that  $am - bn = \alpha$  and  $cm - dn \geq \beta$ , because  $a$  and  $b$  are relatively prime; this would satisfy the inequalities on the left but not on the right. (More precisely, we could use Euclid's algorithm to find integers  $a'$  and  $b'$  such that  $aa' - bb' = 1$ . Then the values  $(m, n) = (\alpha a' + bx, \alpha b' + ax)$  would satisfy the left inequalities but not the right, for infinitely many integers  $x$ , because  $ad - bc \neq 0$ .)

Thus  $R(\alpha, \beta) \equiv R(\alpha', \beta')$  implies that  $\alpha - \alpha' = ka - lb$  and  $\beta - \beta' = kc - ld$ . The converse is trivial.  $\square$

Let  $k$  and  $l$  be integers such that  $\alpha = ka - lb$ . The lemma tells us that  $R(\alpha, \beta) \equiv R(0, \beta - kc + ld)$ ; hence every digitized region  $R(\alpha, \beta)$  has the same shape as some digitized region  $R(\alpha', \beta')$  in which  $\alpha' = 0$ .

It remains to count the inequivalent regions  $R(0, \beta)$  when  $\beta$  is an integer. According to the lemma we have  $R(0, \beta) \equiv R(0, \beta')$  if and only if there exist integers  $(k, l)$  with  $0 = ka - lb$  and  $\beta - \beta' = kc - ld$ . But  $ka = lb$  if and only if  $k = bx$  and  $l = ax$  for some integer  $x$ ; hence the condition reduces to  $\beta - \beta' = bxc - axd = x(bc - ad)$ . In other words,  $R(0, \beta) \equiv R(0, \beta')$  if and only if  $\beta - \beta'$  is a multiple of  $ad - bc$ . The number of inequivalent regions is therefore  $|ad - bc|$ , as claimed.

To complete the proof of the theorem, we must also verify that each of the equivalence classes is equally likely to be the class of the digitized angular region, when the intersection point  $(x_0, y_0)$  is chosen at random in the plane. The notational change from  $(x_0, y_0)$  to  $(\alpha, \beta)$  maps equal areas into equal areas; so we want to prove that the equivalence class of  $R(\alpha, \beta)$  is uniformly distributed among the  $|ad - bc|$  possibilities, when the real numbers  $(\alpha, \beta)$  are chosen at random. Choosing real numbers  $(\alpha, \beta)$  at random leads to uniformly distributed pairs of integers  $([\alpha], [\beta])$ . And if  $[\alpha]$  has any fixed value and  $[\beta]$  runs through all integers, the equivalence class of  $R(\alpha, \beta)$  runs cyclically through all  $|ad - bc|$  possibilities.

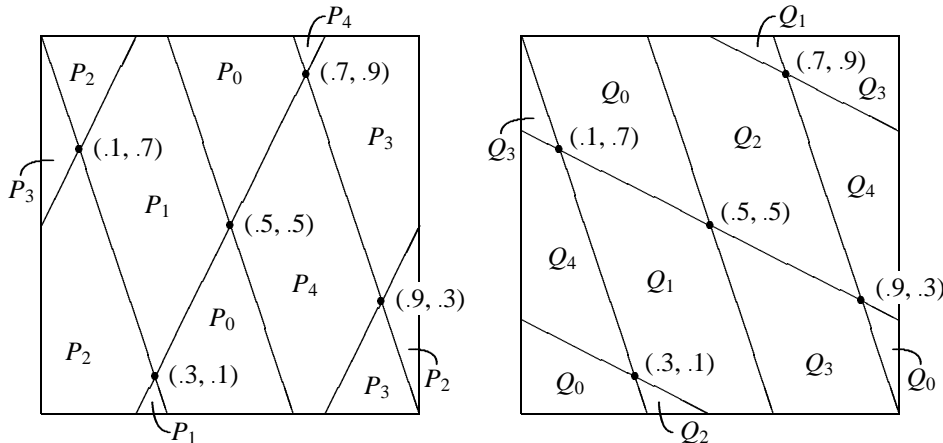
Q.E.D.  $\square$

A close inspection of this proof shows that we can give explicit formulas for the sets of intersection points  $(x_0, y_0)$  that produce equivalent shapes. Let  $D = |ad - bc|$  and let  $R_j$  denote the shape corresponding to region  $R(0, j)$  in the proof, where  $0 \leq j < D$ . Then the digitized angle will have shape  $R_j$  if and only if  $(x_0, y_0)$  lies in the parallelogram whose corners are  $(\frac{1}{2} - bj/D, \frac{1}{2} - aj/D)$  plus

$$((b - d)/D, (a - c)/D), \quad (-d/D, -c/D), \quad (b/D, a/D), \quad (0, 0),$$

or in a parallelogram obtained by shifting this one by an integer amount  $(m, n)$ .

In the special case  $a/b = 2/1$  and  $c/d = -3/1$ , the shapes  $R_j$  are what we called  $P_j$  above; in the special case  $a/b = 3/(-1)$  and  $c/d = -1/2$ , the  $R_j$  are what we called  $Q_j$ . The shapes that appear in the digitized angles depend on the values of  $(x_0 \bmod 1, y_0 \bmod 1)$  in the unit square, according to the following diagrams:



Notice that the parallelograms “wrap around” modulo 1, each taking up an area of  $1/5$ .

When the slope of either line forming an angle is irrational, the number of possible shapes is infinite (indeed, uncountable). But we can still study such digitizations by investigating the shape only in the immediate neighborhood of the intersection point; after all, those pixels are the most critical for human perception. For example, exercises 24.7–9 of *The METAFONT book* [3] discuss the proper way to adjust the vertices of an equilateral triangle so that it will digitize well.

The moral of this story, assuming that stories ought to have a moral, is probably this: If you want to bisect an angle in such a way that both halves of the bisected angle are visually equivalent, then the line of bisection should be such that reflections about this line always map pixels into pixels. Thus, the bisecting line should be horizontal or vertical or at a 45° diagonal, and it should pass through pixel corners and/or pixel centers. Furthermore, your line-rendering algorithm should produce symmetrical results about the line of reflection (see Reference [1]).

This subject is clearly ripe for a good deal of further investigation.

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